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Tilted Squares, Irrational Numbers, and the Pythagorean Theorem

Can you draw a square on a grid encompassing exactly 10 square units? Can you draw a line segment that is exactly $\sqrt{2}$ units long?

In our geometry classes, a line segment with length $\sqrt{2}$ became just as real as one with length 3. Drawing a 10-square-unit diagram was not difficult. Our students tilted squares and counted areas, became frustrated and bored with counting, and looked for patterns so that their calculations would be less tedious and more precise. They made connections between the length of a square's side and its area. They realized that $\sqrt{8}$ was the same as two $\sqrt{2}$ (i.e., $2\sqrt{2}$).

They talked about slopes of segments to describe their illustrations. The square root of 61 to my students meant the hypotenuse of a right triangle with legs of 6 and 5. Their mathematics was visual.

When we finally applied the Pythagorean theorem, students thought it was a cakewalk. The Pythagorean theorem was no longer just $a^2 + b^2 = c^2$. They could have easily named it the Smith theorem after a classmate who discovered it while working with others in a group.

In this article, we present four activities that build a foundation for the Pythagorean theorem and that provide experiences rich with discovery about square roots. These activities were inspired by the Connected Mathematics Project's Looking for Pythagoras activity (Lappan et al. 1998) and address the Number and Operations and Geometry standards of *Principles and Standards for School Mathematics* (NCTM 2000).

PREREQUISITES

Before working with these activities, our students found area and slope by counting squares. They were comfortable with the connection between changes in rise and run and the length of a line segment as shown in **figure 1**. Students understood that going up 3 and right 2 or going right 3 and down 2 produce the same segment length. Any segment involving a combination of three and two for its rise and run has the same length.

Students also debated about the

Edited by **Denisse R. Thompson**, thompson@tempest.coedu.usf.edu, mathematics education, University of South Florida, Tampa, and **Gwen Johnson**, gjohnson@coedu.usf.edu, secondary education, University of South Florida. This department's classroom-ready activities may be reproduced by teachers. Teachers are encouraged to submit manuscripts based on successful activities from their own classroom in a format similar to this article. Of particular interest are activities focusing on the Council's Content and Process Standards and Curriculum Focal Points. Send submissions by accessing **mtms.msubmit.net**.

Fig. 1 Although the order of movements is different, the resulting segment is the same length.

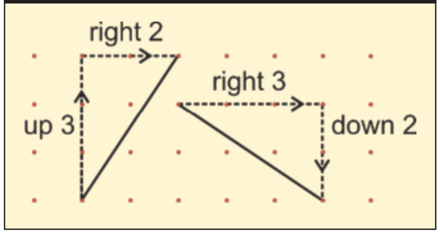


Fig. 2 Tilting the square provided many ways to determine length and area. The tilted square in this example has a side length of $\sqrt{3^2 + 2^2}$, or $\sqrt{13}$, and an area of 13.

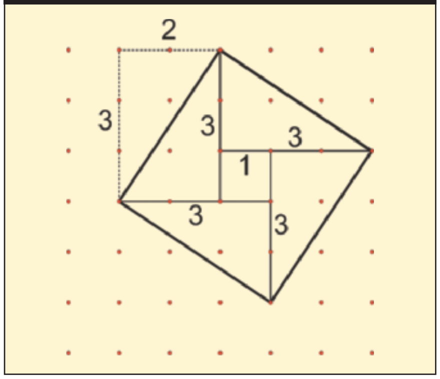


Table 1 Area of tilted squares, with sides of different slope

The Tilt	Area
Up 1, right 1	2
Up 1, right 2	5
Up 1, right 3	10
Up 1, right 4	17
Up 1, right 5	26
Up 1, right 6	37
Up 1, right 7	50
Up 1, right 8	65
Up 1, right 9	82
Up 2, right 2	8
Up 2, right 3	13
Up 2, right 4	20
Etc.	

difference between the lengths of a horizontal or vertical unit compared with a diagonal unit.

One Hundred Squares

To begin, students were given the task found on **activity sheet 1**, in which they were asked to draw as many squares with areas from 1 to 100 as possible. Students found the ten most obvious squares: 1 ∂ 1, 2 ∂ 2, 3 ∂ 3, . . . , 10 ∂ 10. For some, this task took most of the hour; for others, it took a few minutes. Most students decided that they had completed the task and that there were no other squares. We said simply, "There are more."

As soon as students realized that they could tilt a square, they began finding other squares and their areas. (See **fig. 2**.) We wanted our students to tilt the square and count the internal squares, then decide if the area was within the criteria. Many students literally counted the area.

We anticipated that students would get bogged down with the tedious job of counting squares to find areas and ask, "Isn't there an easier way to do this?" Ultimately, we wanted each student to make an organized list and draw as many squares as they could on their own. (See **table 1**'s example of one student's organized list.)

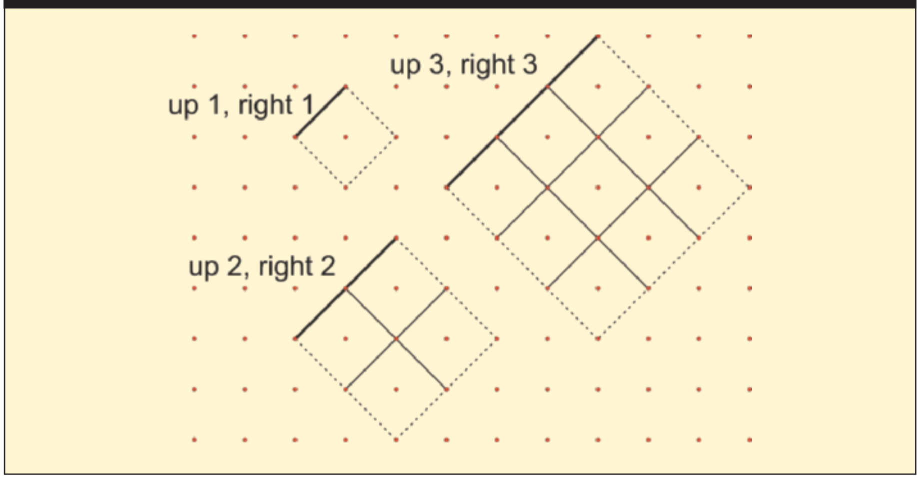
Some students used a scaling method to organize the data by drawing a square as 1 up and 1 right, as 2 up and 2 right, and as 3 up and 3 right. In so doing, they noticed patterns like those in the diagrams in **figure 3**.

Students extended this strategy and knew they could make the squares listed in **table 2**. Then they might look at a square whose side has an 1-up-and-2-right length and find that its area is five. They doubled the pattern to obtain the squares and pattern in **table 3**. The first time we introduced this activity, students worked on it individually. Nearly half gave up. Later, we gave the challenge to groups of three students. Working in a group encouraged them to draw on their own strengths and the strengths of their group members. Ultimately, the challenge became less tedious and frustrating. The activity still begged the question, "Isn't there an easier way to do this?" This scenario set up the discovery of square roots.

SIDES OF SQUARES ARE SQUARE ROOTS

"How does the length of the side of a square relate to the area of the square?" Students answered this question by looking at the nontilted

Fig. 3 The more diagrams students drew, the more students realized that they could construct squares whose areas were not perfect squares.



squares on **activity sheet 1**. If students knew the area of the square, could they find the length of the side?

After gathering data on the nontilted squares, students were challenged to use their calculators. They were to analyze connections between the area of a square to the length of its side. This task naturally led students to discover that the length of any side is the square root of the area.

This is the perfect place to introduce the Pythagorean theorem by using dot paper to relate what students have already completed.

Table 2 Areas of tilted squares, with sides of slope 1

Tilt	1 up, 1 right	2 up, 2 right	3 up, 3 right	4 up, 4 right	5 up, 5 right	6 up, 6 right	7 up, 7 right
Area	2	2 × 4	2 × 9	2 × 16	2 × 25	2 × 36	2 × 49

Table 3 Areas of tilted squares, with sides of slope 1/2

Tilt	1 up, 2 right	2 up, 4 right	3 up, 6 right	4 up, 8 right
Area	5	4 × 5	9 × 5	16 × 5

VARIOUS IRRATIONAL SEGMENT LENGTHS

Part 1 of **activity sheet 2** asks students to determine the length of a given line segment drawn on dot paper. Students either drew or imagined the square whose side is drawn for each line segment. Some students drew the right triangle, with the hypotenuse being the drawn segment. Others determined the first segment length and then saw that the second line segment is two of the first. For example, the first segment on the top left of the sheet has length $\sqrt{2}$. The segment below it is two $\sqrt{2}$ segments put together. Students worked in all directions on this activity. Although some students may struggle, the end result is a better understanding of square roots. Students demonstrate mastery of the Pythagorean theorem and understand that the length of a side of a square is the square root of the area.

When we implemented this activity,

we did not emphasize simplifying radicals, such as

$$2\sqrt{2} = \sqrt{8}.$$

It is important for students to be able to write segment lengths so that they see the connections that are found among the area of a square, the Pythagorean theorem, and patterns. Allow several students to share with the class how they found these lengths and the various ways they found to write each one.

Part 2 of **activity sheet 2** asks students to draw line segments of various rational and irrational lengths and make final connections. For some students, this task is easy. For others, it is difficult. We gave students a sheet of dot paper and asked them to draw line segments with lengths

$$2, 4, 5, 7, 8, 9, 10, 18, 20,$$
$$\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \sqrt{5}, \sqrt{10},$$
$$2\sqrt{5}, \sqrt{8}, \sqrt{9}, \sqrt{18}, \text{ and } \sqrt{20}.$$

The endpoints of the segments had to be on dots, and students had to produce the drawings. We wanted students to be able to both master the concept *and* be able to apply their knowledge. For example, many students know that perpendicular lines have slopes that are opposite reciprocals. However, students often cannot draw two perpendicular lines on dot paper because they do not use the rule that the two slopes are opposite reciprocals. An "a-ha" moment occurs when students can see and realize the connection between the diagram and the fact.

When students learn that $\sqrt{2}$ refers to the length of a side of a square with area 2, ordering rational and irrational numbers is no longer a problem. Our geometry students even began to understand why

$$2\sqrt{2} = \sqrt{8}.$$

They saw how two segments of length $\sqrt{2}$ were equal to the length of a side of a square with area 8 and that, visually, a $\sqrt{8}$ segment looked like two $\sqrt{2}$ segments. These numbers were no longer a mystery. This activity prepares students for the algebraic concept of simplifying radicals and strengthens their number sense of irrational numbers.

FINAL THOUGHTS

The best way to understand a concept is to start with the concrete and build to the abstract. Self-discovery leads to solid understanding. Students often struggle with slope and square roots. These activities ask students to draw squares—a basic shape. Doing so solidifies their understanding of slope and length. The activities helped students become confident in describing lines after tilting a square. They were given a unique connection between slope and rate of change, and how these concepts relate to parallel and perpendicular lines. Throughout the activities, students graphed and communicated their discoveries.

Irrational numbers seem to be a mystery to students. If a calculator displays

$$\sqrt{2} = 1.41421356,$$

students want to write 1.41. However, in our classes, the issue of approximating radicals disappeared, and no calculators were necessary. Students became comfortable with and confident talking about and drawing segments whose lengths were irrational numbers.

This set of activities is demanding, fun, frustrating, and invigorating. Students will walk away feeling that they have tackled something that their older friends do not understand.

REFERENCES

Lappan, Glenda, James T. Fey, William M. Fitzgerald, and Catherine Anderson. *Looking for Pythagoras: The Pythagorean Theorem.* Teacher's Edition, Connected Mathematics Project. Palo Alto, CA: Dale Seymour Publications, 1998.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics.* Reston, VA: NCTM, 2000.

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The solutions to the activity sheets are appended to the online version of "Math Explorations" at www.nctm.org/mtms.

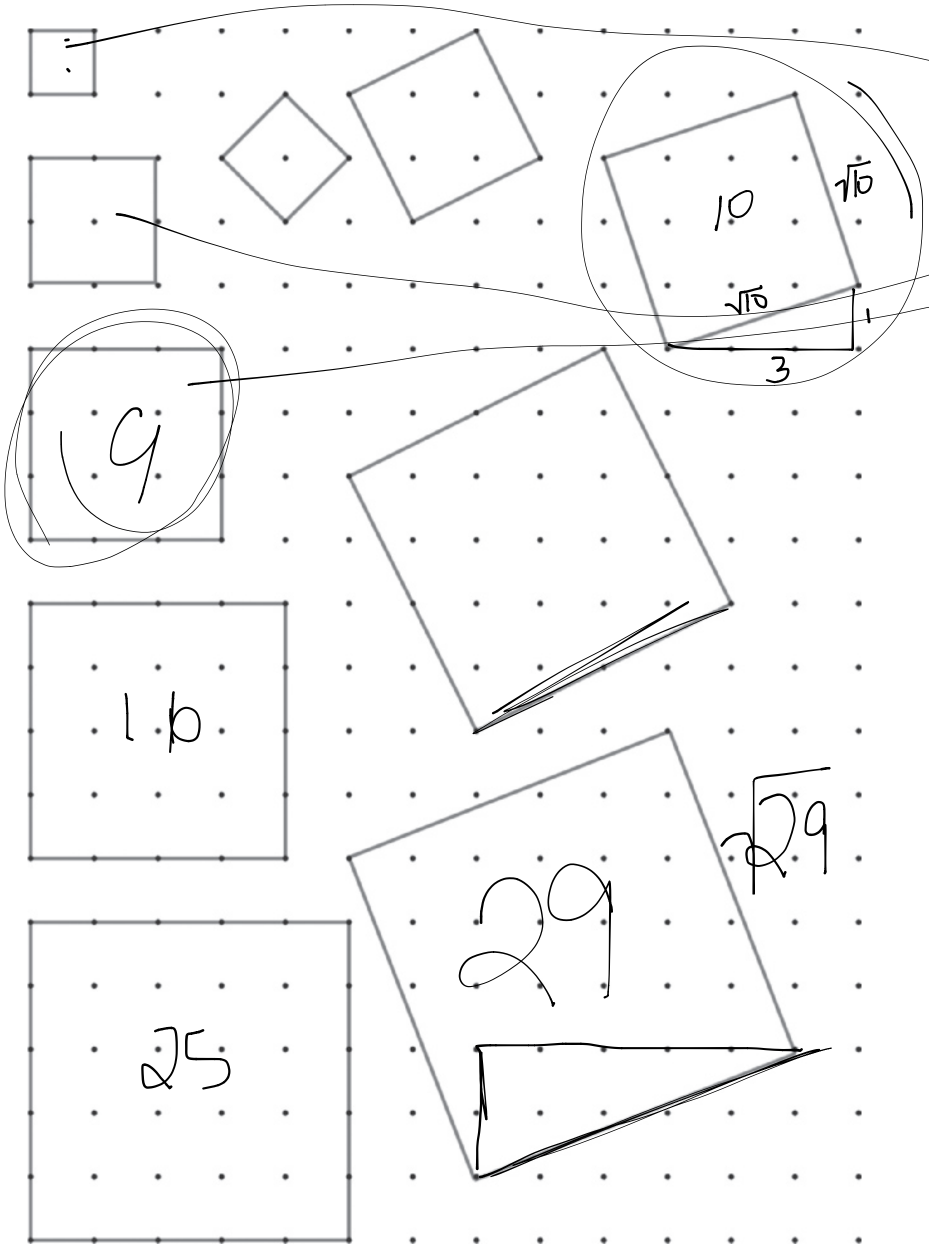
activity sheet 1

Name ______________________________

One Hundred Squares

Part 1
On a separate sheet of dot paper, draw as many squares with areas between 1 and 100 as possible, where a horizontal or vertical line segment between two consecutive dots has a length of 1. Each corner of the square must be on a dot.

Part 2
Complete the table, using the squares on this page, if necessary.



Area	Length of a Side
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
121	11
2	$\sqrt{2}$
5	$\sqrt{5}$
10	$\sqrt{10}$
20	$\sqrt{20}$ $2\sqrt{5}$
29	$\sqrt{29}$
13	$\sqrt{13}$
52	$\sqrt{52}$

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activity sheet 2

Name ______________________________

Various Irrational Segment Lengths

Using your work on the previous activities, determine the length of each of these segments. If possible, find a second way to express the length.

Part 1

a. ______

b. ______

c. ______

d. ______

e. ______

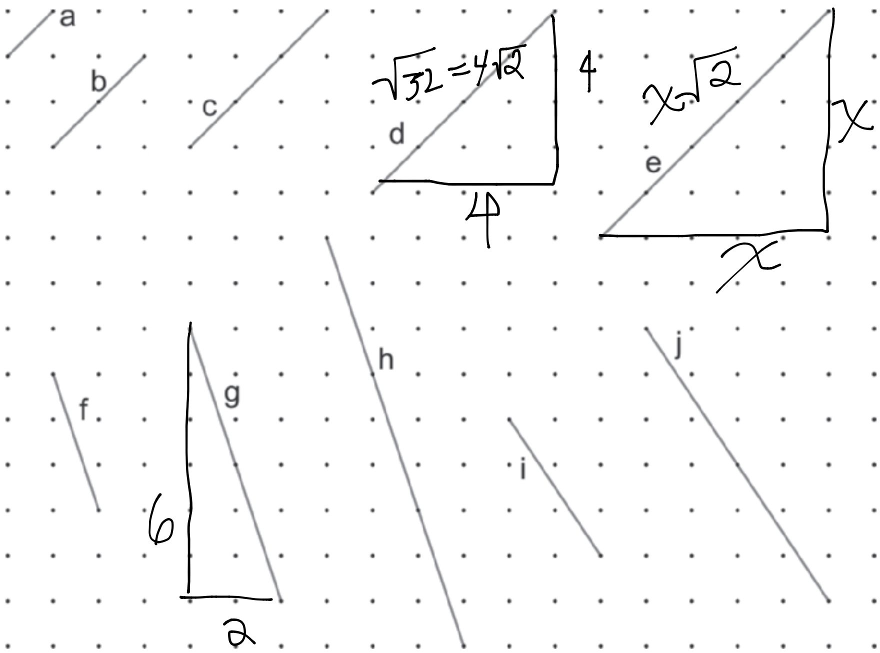
f. ______

g. ______

h. ______

i. ______

j. ______



Part 2

a. ______

b. ______

c. ______

d. ______

e. ______

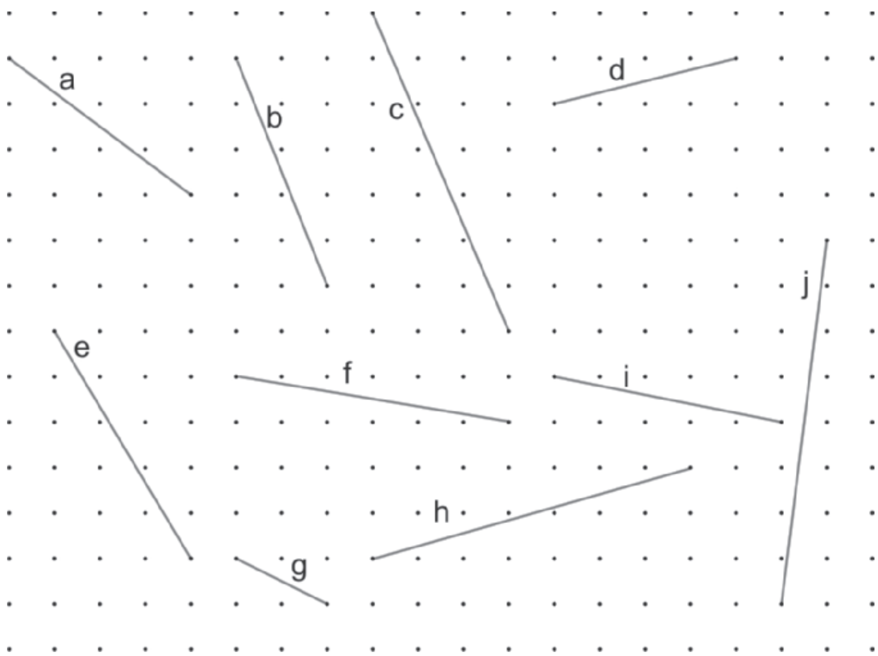
f. ______

g. ______

h. ______

i. ______

j. ______



solutions to mathematical explorations

(Continued from pages 57–62)

ACTIVITY SHEET 1

Part 1

Students will likely initially build squares with horizontal and vertical sides. With coaxing, as described in the article, students will start to draw "tilted" squares, as shown in part 2.

Part 2

Area	Length of a Side
1	1
4	2
9	3
16	4
25	5
36	6
49	7
64	8
121	11
2 (a possible tilt is "1 up and 1 right")	$\sqrt{2}$
5 (a possible tilt is "2 up and 1 right")	$\sqrt{5}$
10 (a possible tilt is "3 up and 1 right")	$\sqrt{10}$
20 (a possible tilt is "4 up and 2 right")	$\sqrt{20}$ or $2\sqrt{5}$
29 (a possible tilt is "5 up and 2 right")	$\sqrt{29}$
13 (a possible tilt is "2 up and 3 right")	$\sqrt{13}$
52 (a possible tilt is "4 up and 6 right")	$\sqrt{52}$

ACTIVITY SHEET 2

Part 1

a. $\sqrt{2}$
b. $\sqrt{8} = 2\sqrt{2}$
c. $\sqrt{18} = 3\sqrt{2}$
d. $\sqrt{32} = 4\sqrt{2}$
e. $\sqrt{50} = 5\sqrt{2}$
f. $\sqrt{10}$
g. $\sqrt{40} = 2\sqrt{10}$
h. $\sqrt{90} = 3\sqrt{10}$
i. $\sqrt{13}$
j. $\sqrt{52} = 2\sqrt{13}$

Part 2

a. 5
b. $\sqrt{29}$
c. $\sqrt{58}$
d. $\sqrt{17}$
e. $\sqrt{34}$
f. $\sqrt{37}$
g. $\sqrt{5}$
h. $\sqrt{53}$
i. $\sqrt{26}$
j. $\sqrt{65}$