Illustrative Mathematics

$(x+3)(x-3)$

x^2-9

A-SSE Seeing Dots

$n+4 \cdot n$

$[(n+2)+2] \cdot [(n+2)-2]$

Task

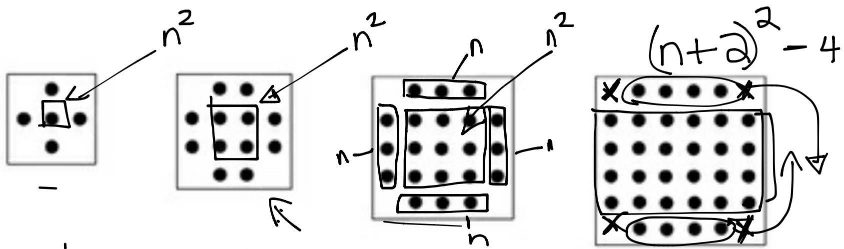
Consider the algebraic expressions below:

	n	2
n	n^2	$2n$
2	$2n$	4

$\boxed{n^2+4n}+\cancel{4-4}$ $(n+2)(n+2)-4$

$$(n+2)^2 - 4 \quad \text{and} \quad n^2 + 4n.$$

a. Use the figures below to illustrate why the expressions are equivalent:



$1=n$ $2=n$ $3=n$ $4=n$

b. Find some ways to algebraically verify the same result.

IM Commentary

The purpose of this task is to identify the structure in the two algebraic expressions by interpreting them in terms of a geometric context. Students are asked to notice a pattern and connect the pattern to the algebraic representation. A potential source of scaffolding is to help students make this connection by labelling the "n-th" figure in the sequence. On the other hand, a potential source of challenge in this task is to encourage a multitude and variety of approaches, both in terms of the geometric argument and in terms of the algebraic manipulation.

Some students might show the equivalence algebraically from the start, either by expanding or by factoring. The algebraic approach should be rewarded, not discouraged. A student who expands could be asked if there is another algebraic method; a student who factors could be asked if there is a way of relating this form to the figure. Indeed, as an alternate solution to the one given in the solution, we observed that the factored form, $(n+4)n$, can be related to the figures as follows: If you take the top and bottom borders, turn them vertically, and place them next to the rest of the figure, you get an $(n+4) \times n$ rectangle of dots.

There is also an opportunity here to discuss the process of justifying an algebraic identity. For one, the algebraic solution in part (b) applies to all real numbers n, whereas the proof by pictures only directly applies to the case that n is a positive integer (though students could be encouraged to replace "numbers of dots" with "areas of regions" to give a version of the geomtric proof that works for all positive real numbers.)

Solution

a. A geometric approach to the problem proceeds by identifying, somewhere in the n-th figure, the value n, and seeing two ways of looking at the dots, giving both $(n+2)^2 - 4$ and $n^2 + 4n$. One such approach (among many) is below.

Let n be the number of the figure, with $n = 1$ at the left. We count the dots in each figure in terms of n in two different ways. One represents $n^2 + 4n$ and the other represents $(n+2)^2 - 4$.

Visualizing $n^2 + 4n$:

- n^2 is the inside full square.
- $4n$ is the four outside borders with n in each.

Visualizing $(n+2)^2 - 4$:

- Imagine the larger square with the four additional dots filled in at the corners. Then $(n+2)^2$ is the number of dots in the larger square.
- 4 is the number of dots added.

b. Perhaps most directly, we have

$$(n+2)^2 - 4 = (n^2 + 4n + 4) - 4 = n^2 + 4n.$$

Alternatiely, reversing the steps in this series of equalities is precisely the process of completing the square for the expression $n^2 + 4n$. Similarly, the left-hand side could be viewed as a difference of two squares, in which case we can reason:

$$(n+2)^2 - 4 = ((n+2)+2)((n+2)-2) = (n+4)(n) = n^2 + 4n.$$